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Positive energy representations, holomorphic discrete series and finite-dimensional irreps

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Abstract

Let G be a semi-simple non-compact Lie group with unitary lowest/highest weight representations. We consider explicitly the relation among three types of representations of G : positive energy (unitary lowest weight) representations, (holomorphic) discrete series representations and non-unitary finite-dimensional irreps. We consider mainly the conformal groups $SO_o(n, 2)$ treating in full detail the cases $n = 1, 3, 4$.

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1. Introduction

Let G be a semi-simple non-compact Lie group with unitary lowest/highest weight representations, i.e., (G, K) is a Hermitian symmetric pair, where K is a maximal compact subgroup of G [1]. Let \mathcal{G} be the Lie algebra of G . Then, \mathcal{G} is one of the following Lie algebras: $su(m, n)$, $so(n, 2)$, $sp(2n, R)$, $so^*(2n)$, $E_{6(-14)}$, $E_{7(-25)}$ [2]. These groups/algebras also have discrete series representations, since $\text{rank } G = \text{rank } K$, and $G \supset K \supset H$, where H is a Cartan subalgebra of G [3].

In this paper we start a discussion on the relation among three types of representations of G : positive energy (i.e., unitary lowest weight) representations, (holomorphic) discrete series representations and finite-dimensional representations (the latter are not unitary).

There are some general facts that are known about these relationships. For example every discrete series representation has the same infinitesimal character (Casimirs) as some finite-dimensional representations [3]. According to the results of [1, 3, 4] a holomorphic/antiholomorphic discrete series representation is (infinitesimally) equivalent to a unitary lowest/highest weight representation. The submerging of the set of discrete points enumerating the holomorphic discrete series in the semi-infinite interval parametrizing the unitary lowest weight representations is given in [4].² The embedding of discrete series

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² Note that EHW [4] work in the conjugate picture with highest weight modules.

representations into elementary representations³ (also called generalized principal series representations) is given in [5–7]. (Other pertinent references are [8–13].) However, the latter two very important connections are not used simultaneously in the mathematical literature.

Our input in the discussion of these relationships may be summarized as follows. First, we use all relationships among the three types of representations. In particular, we use essentially the fact that discrete series representations and finite-dimensional representations occur as subrepresentations of elementary representations. The elementary representations in question are topologically reducible (and not unitary). We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [14, 15]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators⁴. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation. Especially important are the multiplets containing (as subrepresentation of some reducible ER of the multiplet) a finite-dimensional representation. Each such multiplet contains some discrete series representation(s) (as subrepresentation(s) of other reducible ER(s) of the multiplet), and all discrete series representations are contained in some such multiplet. Furthermore, from these multiplets—by certain limiting procedure—one can obtain all multiplets containing limits of discrete series. (The latter resulting multiplets do not contain finite-dimensional representations.) Finally, using the multiplets, we can identify the intertwining operators relevant for discrete series representations and the finite-dimensional representation.

We should also mention that in distinction to mathematicians, in our considerations we are using induction also from maximal parabolics that are not cuspidal (e.g., on the example of $\mathcal{G} = so(n, 2)$ in this paper).

In the present paper we start such a description by discussing in some detail the conformal case when $\mathcal{G} = so(n, 2)$. There are two typical cases when $n > 2$: n odd and n even. Furthermore, the case $n = 1$ is special and is discussed separately. (The case $n = 2$ is reduced to the case $n = 1$.) Thus, the paper is organized as follows. In section 2 we give the general setting. In section 3 we specify the setting to the conformal case. In sections 4–6 we treat the cases $n = 1, 3, 4$ in detail.

2. Preliminaries

Let G be a semisimple non-compact Lie group and K a maximal compact subgroup of G . Then we have an Iwasawa decomposition $G = KAN$, where A is abelian simply connected, a vector subgroup of G , N is a nilpotent simply connected subgroup of G preserved by the action of A . Further, let M be the centralizer of A in K . Then the subgroup $P_0 = MAN$ is a minimal parabolic subgroup of G . A parabolic subgroup $P = M'A'N'$ is any subgroup of G (including G itself) which contains a minimal parabolic subgroup⁵.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [17]. For the classification of all irreducible representations it is enough to use only the so-called *cuspidal* parabolic subgroups $P = M'A'N'$, singled out by the condition that $\text{rank } M' = \text{rank } M' \cap K$ [18, 19], so that M' has discrete series representations [3]. However, often induction from non-cuspidal parabolics is also convenient and we shall use it below.

³ The precise definition with relevant references is given below.

⁴ For simplicity only the operators which are not compositions of other operators are depicted. The ERs which are related by non-trivial intertwining operators are said to be partially equivalent.

⁵ The number of non-conjugate parabolic subgroups is 2^r , where $r = \text{rank } A$, cf, e.g., [16].

Let ν be a (non-unitary) character of A' , $\nu \in \mathcal{A}'^*$, let μ fix an irreducible representation D^μ of M' on a vector space V_μ .

We call the induced representation $\chi = \text{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [20]. (These are called *generalized principal series representations* (or *limits thereof*) in [6].) Their spaces of functions are

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) | \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\mathcal{F}(g)\}, \quad (2.1)$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$.

For our purposes we need to restrict to *maximal* parabolic subgroups P , (so that $\text{rank } A' = 1$), that may not be cuspidal (the importance of such occurrences is explained on the example of $\mathcal{G} = so(n, 2)$ below). For the representations that we consider the character ν is parameterized by a real number d , called the conformal weight or energy (the latter for reasons that will become clear below).

We also restrict to the case of finite-dimensional (nonunitary) representations μ of M' .

An important ingredient in our considerations are the *unitary lowest weight representations* of \mathcal{G} . These can be realized as factor modules of Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, the lowest weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [15]. Unitarity means positivity with respect to the Shapovalov form in which the conjugation is the one singling out \mathcal{G} from $\mathcal{G}^{\mathbb{C}}$.

Actually, since our ERs are induced from finite-dimensional representations of M' the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* \tilde{V}^Λ such that the role of the lowest weight vector v_0 is taken by the finite-dimensional space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d . Matters are arranged so that there is a real number d_0 called the *first reduction point* (FRP), such that for $d > d_0$ the GVMs are irreducible and unitary. For $d = d_0$ the GVM is reducible (in general) with invariant subspace I^Λ so that the factor space $L_\Lambda \cong \tilde{V}^\Lambda / I^\Lambda$ is irreducible and unitary. For $d < d_0$ in some cases there is a discrete set of d values for which the GVM is reducible and again the factor space L_Λ is irreducible and unitary. This picture was known for the conformal cases $\mathcal{G} = so(n, 2)$ when $n = 3$ [21] (for more modern exposition cf also [22]) and $n = 4$ [23], but was established for all algebras with lowest/highest weight modules in [4].⁶

We turn now to the relation of discrete series representations with lowest weight modules.

The unitary lowest weight generalized Verma modules are infinitesimally equivalent to *holomorphic discrete series* when $d = d_0 + kc_0$, $k = A(\lambda_0) + 1, 2, \dots, c_0$, $A(\lambda_0) \in \mathbb{N}$, [4]. The GVMs with $d = d_0 + c_0 A(\lambda_0)$ are infinitesimally equivalent to the so-called *limits of discrete series* [4]. The latter are not related to finite-dimensional representations.

The irreps for $d > d_0$ are also called analytic continuation of the discrete series.

In order to be more specific, in the following sections we consider the conformal cases.

3. Conformal groups

Let $G = SO_o(n, 2)$. We shall consider first the case $n > 2$. Then G is the conformal group in n -dimensional Minkowski spacetime. The Lie algebra, i.e., the conformal algebra $\mathcal{G} = so(n, 2)$ has three non-trivial non-conjugate parabolic subalgebras $\mathcal{P}_i = \mathcal{M}_i \oplus \mathcal{A}_i \oplus \mathcal{N}_i$, $i = 0, 1, 2$, where

⁶ Note that EHW [4] work with highest weight modules, thus, their ranges are limited from above, while we work with lowest weight modules and our ranges are limited from below. The latter is done to have the intuitive picture of positive energy spectrum bounded from below. There is also a shift of the initial points—in our case d —as energy—is positive, while in the notation of [4] the spectrum includes the point zero.

$$\mathcal{M}_0 \cong so(n-2), \quad \mathcal{A}_0 \cong so(1,1) \oplus so(1,1), \quad \mathcal{N}_0 \cong \mathbb{R}^{2n-2} \quad (3.1a)$$

$$\mathcal{M}_1 \cong so(n-2) \oplus so(2,1), \quad \mathcal{A}_1 \cong so(1,1), \quad \mathcal{N}_1 \cong \mathbb{R}^{2n-3} \quad (3.1b)$$

$$\mathcal{M}_2 \cong so(n-1,1), \quad \mathcal{A}_2 \cong so(1,1), \quad \mathcal{N}_2 \cong \mathbb{R}^n \quad (3.1c)$$

The parabolic \mathcal{P}_0 is a minimal one, and thus is cuspidal. The other two parabolics are maximal, \mathcal{P}_1 is also cuspidal, while \mathcal{P}_2 is cuspidal only if n is odd.

We shall use representations induced from the parabolic \mathcal{P}_2 since the sets of finite-dimensional (nonunitary) representations of \mathcal{M}_2 are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $so(n)$ which is the semi-simple subalgebra of the maximal compact subalgebra $\mathcal{K} = so(n) \oplus so(2)$. Thus, these induced representations are representations of finite \mathcal{K} -type [1]. Relatedly, the number of ERs in the corresponding multiplets is equal to $|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|/|W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| = 2(1 + \tilde{h})$, $\tilde{h} \equiv \lfloor \frac{n}{2} \rfloor$, where \mathcal{H} is a Cartan subalgebra of both \mathcal{G} and \mathcal{K} . Note also that $\mathcal{K}^{\mathbb{C}} \cong \mathcal{M}_2^{\mathbb{C}} \oplus \mathcal{A}_2^{\mathbb{C}}$.

The Bruhat decomposition [24] of \mathcal{G} which corresponds to this parabolic is

$$\mathcal{G} = \tilde{\mathcal{N}}_2 \oplus \mathcal{M}_2 \oplus \mathcal{A}_2 \oplus \mathcal{N}_2, \quad (3.2)$$

where $\tilde{\mathcal{N}}_2 = \theta \mathcal{N}_2$ (θ is the Cartan involution in \mathcal{G}). The subalgebras in this decomposition have direct physical meaning: \mathcal{M}_2 is the Lorentz algebra of n -dimensional Minkowski spacetime M^n (the latter differs from \mathbb{R}^n by the Lorentzian metric), \mathcal{N}_2 is the translation algebra of M^n , \mathcal{A}_2 is the subalgebra of dilatations, $\tilde{\mathcal{N}}_2$ is the subalgebra of special conformal transformations of M^n .

We label the signature of the ERs of \mathcal{G} as follows:

$$\begin{aligned} \chi &= \{n_1, \dots, n_{\tilde{h}}; c\}, & n_j &\in \mathbb{Z}/2, & c &= d - \frac{n}{2}, \\ |n_1| &< n_2 < \dots < n_{\tilde{h}}, & n &\text{even}, & \\ 0 \leq n_1 &< n_2 < \dots < n_{\tilde{h}}, & n &\text{odd}, \end{aligned} \quad (3.3)$$

where the last entry of χ labels the characters of \mathcal{A}_2 , and the first \tilde{h} entries are labels of the finite-dimensional nonunitary irreps of \mathcal{M}_2 (or of the finite-dimensional unitary irrep of $so(n)$), which also fulfil the requirement that the n_i 's are either all integer, or all half-integer.

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simple intuitive way:

$$\begin{aligned} \chi_1^{\pm} &= \{\epsilon n_1, \dots, n_{\tilde{h}}; \pm n_{\tilde{h}+1}\}, & n_{\tilde{h}} &< n_{\tilde{h}+1}, \\ \chi_2^{\pm} &= \{\epsilon n_1, \dots, n_{\tilde{h}-1}, n_{\tilde{h}+1}; \pm n_{\tilde{h}}\} \\ \chi_3^{\pm} &= \{\epsilon n_1, \dots, n_{\tilde{h}-2}, n_{\tilde{h}}, n_{\tilde{h}+1}; \pm n_{\tilde{h}-1}\} \\ &\dots \\ \chi_{\tilde{h}}^{\pm} &= \{\epsilon n_1, n_3, \dots, n_{\tilde{h}}, n_{\tilde{h}+1}; \pm n_2\} \\ \chi_{\tilde{h}+1}^{\pm} &= \{\epsilon n_2, \dots, n_{\tilde{h}}, n_{\tilde{h}+1}; \pm n_1\} \\ \epsilon &= \begin{cases} \pm, & \text{for } n \text{ even} \\ 1, & \text{for } n \text{ odd.} \end{cases} \end{aligned} \quad (3.4)$$

The ERs in the multiplet are related by intertwining integral and differential operators. The integral operators were introduced by Knapp and Stein [25]. These operators intertwine the pairs $\tilde{\mathcal{C}}_i^+$ and $\tilde{\mathcal{C}}_i^-$:

$$G_i^+ : \tilde{\mathcal{C}}_i^- \longrightarrow \tilde{\mathcal{C}}_i^+, \quad G_i^- : \tilde{\mathcal{C}}_i^+ \longrightarrow \tilde{\mathcal{C}}_i^-, \quad i = 1, \dots, 1 + \tilde{h}. \quad (3.5)$$

Matters are arranged so that in every multiplet only the ER with signature χ_1^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace \mathcal{E} . The latter corresponds to the finite-dimensional unitary irrep of $so(n+2)$ with signature $\{n_1, \dots, n_{\tilde{h}}, n_{\tilde{h}+1}\}$. The subspace \mathcal{E} is annihilated by the operator G_1^+ and is the image of the operator G_1^- .

Analogously, in every multiplet only the ER with signature χ_1^+ contains holomorphic discrete series representation. In fact, it also contains the conjugate anti-holomorphic discrete series. The direct sum of the holomorphic and the antiholomorphic representations are realized in an invariant subspace \mathcal{D} of the ER χ_1^+ . That subspace is annihilated by the operator G_1^- , and is the image of the operator G_1^+ .

Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

The intertwining differential operators correspond to non-compact positive roots of the root system of $so(n+2, \mathbb{C})$, cf [15]. (In the current context, compact roots of $so(n+2, \mathbb{C})$ are those that are roots also of the subalgebra $so(n, \mathbb{C})$, the remainder of the roots are non-compact.) Let us denote by $\tilde{\mathcal{C}}_i^\pm$ the representation space with signature χ_i^\pm . The intertwining differential operators act as follows:

$$\begin{aligned} d_i : \tilde{\mathcal{C}}_i^- &\longrightarrow \tilde{\mathcal{C}}_{i+1}^-, & i = 1, \dots, \tilde{h}, & \forall n \\ d'_i : \tilde{\mathcal{C}}_{i+1}^+ &\longrightarrow \tilde{\mathcal{C}}_i^+, & i = 1, \dots, \tilde{h}, & \forall n \\ d_{\tilde{h}} &= d'_{\tilde{h}}, & n \text{ even} & \\ d_{\tilde{h}+1} : \tilde{\mathcal{C}}_{\tilde{h}+1}^- &\longrightarrow \tilde{\mathcal{C}}_{\tilde{h}}^+, & n \text{ even} & \\ d_{\tilde{h}+1} : \tilde{\mathcal{C}}_{\tilde{h}}^- &\longrightarrow \tilde{\mathcal{C}}_{\tilde{h}+1}^+, & n \text{ even.} & \end{aligned} \tag{3.6}$$

The degrees of these intertwining differential operators are given just by the differences of the c entries:

$$\begin{aligned} \deg d_i &= \deg d'_i = n_{\tilde{h}+2-i} - n_{\tilde{h}+1-i}, & i = 1, \dots, \tilde{h}, & \forall n \\ \deg d_{\tilde{h}+1} &= n_2 + n_1, & n \text{ even.} & \end{aligned} \tag{3.7}$$

The multiplets can be seen pictorially in [26].⁷ The equalities between some intertwining differential operators for n even in (3.6) is due to the fact that these operators are produced by singular vectors corresponding to the same positive roots of the root system of $so(n+2, \mathbb{C})$, cf [15].

More explicitly, for n -even, $n = 2\ell$, the root system of $so(n+2, \mathbb{C})$ may be given by vectors $e_i \pm e_j$, $\ell + 1 \geq i > j \geq 1$, where e_i form an orthonormal basis in $\mathbb{R}^{\ell+1}$, i.e., $(e_i, e_j) = \delta_{ij}$. The non-compact roots may be taken as $e_{\ell+1} \pm e_i$. The roots $e_{\ell+1} - e_i$, $2 \leq i \leq \ell$, correspond to the operators $d_{\ell+1-i}$, the roots $e_{\ell+1} + e_i$, $2 \leq i \leq \ell$, correspond to the operators $d'_{\ell+1-i}$, the roots $e_{\ell+1} \pm e_1$ correspond to the operators d_ℓ, d'_ℓ , resp.

For n -odd, $n = 2\ell + 1$, the root system of $so(n+2, \mathbb{C})$ may be given by vectors $e_i \pm e_j$, $\ell + 1 \geq i > j \geq 1$, e_k , $1 \leq k \leq \ell + 1$. The non-compact roots may be taken as $e_{\ell+1} \pm e_i, e_{\ell+1}$. The roots $e_{\ell+1} - e_i$, $1 \leq i \leq \ell$, correspond to the operators $d_{\ell+1-i}$, the roots $e_{\ell+1} + e_i$, $1 \leq i \leq \ell$, correspond to the operators $d'_{\ell+1-i}$. The root $e_{\ell+1}$ has a special position, for $n_i \in \mathbb{N}$ it corresponds to differential operators of degree $2n_i$ which are degenerations of the integral operators G_i^+ . The latter phenomenon is given explicitly for $n = 3$ in [22].

⁷ Actually the diagrams in [26] are for the corresponding Euclidean cases $so(n+1, 1)$ —the diagrams are the same for both signatures.

Another parametrization of the ER/GVM is by the so-called Dynkin labels. These are defined as follows:

$$m_i \equiv (\Lambda + \rho, \alpha_i^\vee), \tag{3.8}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of \mathcal{G} , α_i denotes the simple roots of \mathcal{G} , $\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i, \alpha_i)$ is the co-root of α_i . Often it is convenient to consider the so-called Harish-Chandra parameters:

$$m_\alpha \equiv (\Lambda + \rho, \alpha^\vee), \tag{3.9}$$

where α is any positive root of \mathcal{G} . These parameters are redundant, since obviously they can be expressed in terms of the Dynkin labels, however, as we shall see below, some statements are best formulated in their terms.

The numbers $\deg d_i, \deg d'_i$, are actually part of the Harish-Chandra parameters which correspond to the non-compact positive roots of $so(n + 2, \mathbb{C})$. From these, only $\deg d_1$, corresponds to a simple root, i.e., is a Dynkin label.

These labellings will be used in the examples $n = 3, 4$ below.

Above we restricted to $n > 2$. The case $n = 2$ is reduced to $n = 1$ since $so(2, 2) \cong so(1, 2) \oplus so(1, 2)$. The case $n = 1$ is special and is treated separately in the next section.

4. SL(2, R)

We start with the conformal case for $n = 1$: $G = SO_o(1, 2) \cong SL(2, \mathbb{R})/\mathbb{Z}_2$. This is treated separately since $r = 1$ and there is only one non-trivial parabolic. Next, the Lie algebra $so(1, 2) \cong sl(2, \mathbb{R})$ is maximally split and the subalgebra \mathcal{M} is trivial. Furthermore, this case is simpler and we can give more details.

4.1. Discrete series and limits thereof

We consider $G = SL(2, \mathbb{R})$ following in this subsection Gelfand *et al* [27] (the original results are in [28, 29]). The elementary representations of $SL(2, \mathbb{R})$ are parametrized by a complex number s and a signature $\epsilon = 0, 1$. We shall denote the ERs by $D_\chi, \chi = [s, \epsilon]$. The discrete unitary series are realized as subspaces of $D_{-s} \equiv D_\chi$, with $\chi = [-s, \epsilon = s \pmod{2}]$, when $s \in \mathbb{Z}_+$. More precisely, there are two discrete series UIRs F_{-s}^\pm , which are invariant subspaces of D_{-s} .

The UIRs realized in F_{-s}^+ nowadays are called *holomorphic discrete series*, they have a lowest weight vector, the UIRs realized in F_{-s}^- nowadays are called *antiholomorphic discrete series*, they have a highest weight vector. Their direct sum $F_{-s} = F_{-s}^+ \oplus F_{-s}^-$ is an invariant subspace of D_{-s} .

For $s \neq 0$ the Casimir (the infinitesimal character) of D_{-s} is equal to the Casimir of D_s , which contains as subrepresentation the finite-dimensional (non-unitary) representation E_s of $SL(2, \mathbb{R})$ of dimension s . The UIRs F_0^\pm are not related to finite-dimensional representations, correspondingly, they do not fulfil Harish-Chandra's criterion for discrete series, and thus, nowadays they are called *limits of discrete series*.

For fixed $s \neq 0$ the ERs D_{-s} and D_s are partially equivalent, which is realized by two integral operators (later introduced in [25] in general): $\mathcal{A}_{\pm s} : D_{\mp s} \longrightarrow D_{\pm s}$, cf (3.5). Thus, these two ERs form a doublet. The operator \mathcal{A}_s annihilates F_{-s} and its image is $E_s \subset D_s$. The operator \mathcal{A}_{-s} annihilates E_s and its image is F_{-s} . Note also that the factor space $F_s = D_s/E_s$ is the direct sum of two further subspaces: $F_s = F_s^+ \oplus F_s^-$, and that the operator \mathcal{A}_{-s} maps F_s^\pm onto F_{-s}^\pm .

For $s = 0$ one has $D_0 = F_0 = F_0^+ \oplus F_0^-$ (setting $E_0 = 0$).

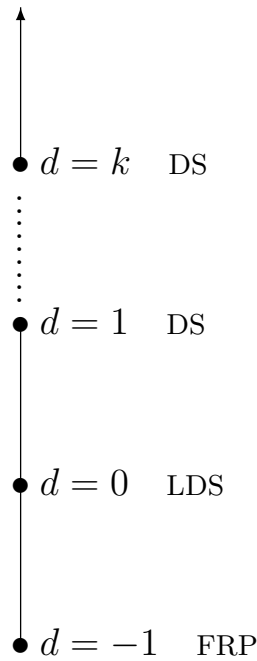


Figure 1. $sl(2, R)$.

4.2. Lowest weight representations

As we mentioned the lowest weight representations are conveniently realized via the lowest weight Verma module V^Λ over the complexification $\mathcal{G}^\mathbb{C} = sl(2, \mathbb{C})$ of the Lie algebra $\mathcal{G} = sl(2, \mathbb{R})$, and the weight $\Lambda = \Lambda(\chi) = \Lambda(s)$ is determined uniquely by χ .

For $s \in \mathbb{N}$ the Verma module $V^{\Lambda(s)}$ is reducible, it has a finite-dimensional factor-representation $E_s^+ = V^{\Lambda(s)} / I^{\Lambda(s)}$ of real dimension s , (which naturally can be identified with E_s from above).

By introducing the conjugation singling out \mathcal{G} we can use the Shapovalov form on $U(\mathcal{G}^+)$ (\mathcal{G}^+ are the raising generators of $\mathcal{G}^\mathbb{C}$) to define a scalar product in V^Λ and then positivity produces the list of lowest weight modules.

Using as parameter $d = -s$ the condition for positivity is $d \geq -1$. The point $d = -1$ is the first reduction point, which happens in the Verma module $V^{\Lambda(1)}$, cf above. There the factor UIR is trivial (being one dimensional). For $d > -1$ the Verma modules $V^{\Lambda(-d)}$ are irreducible and unitarizable. They are called positive energy representations, the parameter d being the energy or conformal weight. They are also called analytic continuation of the discrete series F_{-s}^+ . In particular, for $d = -s \in \mathbb{N}$ the Verma module $V^{\Lambda(-d)}$ is infinitesimally equivalent to the holomorphic discrete series irrep F_{-s}^+ , while for $d = 0$ it is infinitesimally equivalent to the limit of holomorphic discrete series irrep F_0^+ . All this is illustrated in figure 1.

A similar construction exists if we take the conjugate highest weight Verma modules, the role of F_{-s}^+, E_s^+ being played by F_{-s}^-, E_s^- . In that case we are speaking about *negative energy representations*, with parameter $d = s$, so that we have $d \leq 1$.

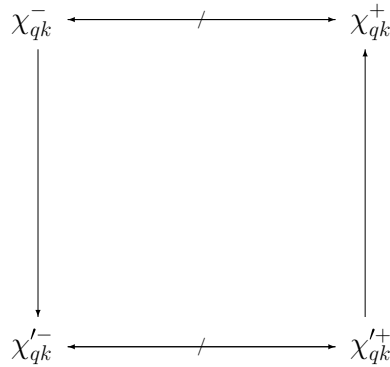


Figure 2. Quartet of partially equivalent ERs and intertwining operators for $so(3, 2)$ and $so(4, 1)$ (arrows are differential operators, dashed arrows are integral operators).

5. $SO(3, 2)$

The algebra $so(3, 2)$ has, besides the minimal parabolic, two maximal (cuspidal) parabolic subalgebras which are isomorphic (though non-conjugate!). So below we fix one of these maximal parabolic subalgebras and denote it \mathcal{P}_{\max} . Here the reducible ERs that have finite \mathcal{K} -type representations can be induced from $\mathcal{P}_{\max} = \mathcal{M}_2 \mathcal{A}_2 \mathcal{N}_2$, where $\mathcal{M}_2 = so(2, 1)$, $\dim \mathcal{A}_2 = 1$, $\dim \mathcal{N}_2 = 3$. Their signatures are given by $\chi = [E_0, s_0]$, where we have introduced the traditionally used energy E_0 and spin s_0 , $s_0 = 0, \frac{1}{2}, 1, \dots$ (the latter parametrizing the finite-dimensional irreps of $so(2, 1)$ or $so(3)$).

Alternatively, the ERs (GVMs) are determined by the two Dynkin labels: $m_i = \langle \Lambda + \rho, \alpha_i \rangle$, $i = 1, 2$, where α_i are the simple roots. The relation between the two parametrizations is

$$\begin{aligned} m_1 &= 2s_0 + 1, & m_2 &= 1 - E_0 - s_0, \\ \chi &= [E_0, s_0] = \left[\frac{1}{2}(3 - m_1) - m_2, \frac{1}{2}(m_1 - 1) \right] \end{aligned} \tag{5.1}$$

The numbers $m_3 = m_1 + 2m_2 = 3 - 2E_0$, $m_4 = m_1 + m_2 = 2 - E_0 + s_0$ correspond to the two non-simple positive roots $\alpha_3 = \alpha_1 + \alpha_2$, $\alpha_4 = 2\alpha_1 + \alpha_2$. The root α_1 is compact, the other—non-compact. The set of the four numbers m_i , $i = 1, 2, 3, 4$, are the Harish-Chandra parameters.

5.1. Multiplets, finite-dimensional irreps and discrete series

The reducible such ERs are grouped in quartets, doublets and singlets [22].

The quartets are depicted in figure 2, cf [22]. The signatures of the ERs of the quartet are given by

$$\begin{aligned} \chi_{q,k}^\pm &= \left[\frac{1}{2}(3 \mp (q + 2k)), \frac{1}{2}(q - 1) \right], \\ \chi'_{q,k}^\pm &= \left[\frac{1}{2}(3 \mp q), \frac{1}{2}(q - 1 + 2k) \right], & q, k &\in \mathbb{N}. \end{aligned} \tag{5.2}$$

The quartets are in 1-to-1 correspondence with the finite-dimensional irreps of G since in each quartet there is exactly one ER which contains (as subrepresentation) a finite-dimensional irrep.

Table 1. Harish-Chandra parameters for $so(3, 2)$: $m_i = \langle \Lambda + \rho, \alpha_i \rangle, i = 1, 2, 3, 4, m_3 = m_1 + 2m_2, m_4 = m_1 + m_2, m_1 \equiv 2s_0 + 1$.

| ER | m_1 | m_2 | m_3 | m_4 |
|---------------------------------|----------|----------------|-----------|----------------|
| $\chi_{q,k}^-$ | q | k | $q + 2k$ | $q + k$ |
| $\chi_{q,k}^+$ | q | $-q - k$ | $-q - 2k$ | $-k$ |
| $\chi_{q,k}^{\prime-}$ | $q + 2k$ | $-k$ | q | $q + k$ |
| $\chi_{q,k}^{\prime+}$ | $q + 2k$ | $-q - k$ | $-q$ | k |
| χ_q^- | q | 0 | q | q |
| χ_q^+ | q | $-q$ | $-q$ | 0 |
| χ_n^s | $2n$ | $-n$ | 0 | n |
| Rac, $\chi_{1, \frac{1}{2}}^-$ | 1 | $\frac{1}{2}$ | 2 | $\frac{3}{2}$ |
| Rac*, $\chi_{1, \frac{1}{2}}^+$ | 1 | $-\frac{3}{2}$ | -2 | $-\frac{1}{2}$ |
| Di, $\chi_{2, -\frac{1}{2}}^-$ | 2 | $-\frac{1}{2}$ | 1 | $\frac{3}{2}$ |
| Di*, $\chi_{2, -\frac{1}{2}}^+$ | 2 | $-\frac{3}{2}$ | -1 | $\frac{1}{2}$ |

The latter are parametrized by the positive integer Dynkin labels which we denote by q, k . The corresponding finite-dimensional irrep is denoted $E_{q,k}$, it has dimension: $qk(q + k)(q + 2k)/6$, [22], and is contained in the ER denoted by $\chi_{q,k}^-$.

Consequently, the quartets also hold the discrete series representations. For fixed q, k the discrete series are contained in $\chi_{q,k}^+, \chi_{q,k}^{\prime+}$, we know that there are two distinct non-conjugate cases of discrete series [6].

The doublets are denoted by χ_q^\pm , and the expression for their signatures can be obtained from the signatures of $\chi_{q,k}^\pm$ or from $\chi_{q,k}^{\prime\pm}$ by setting $k = 0$, i.e.,

$$\chi_q^\pm = \left[\frac{1}{2}(3 \mp q), \frac{1}{2}(q - 1) \right]. \tag{5.3}$$

Note that for all pairs of ERs with signature distinguished by \pm the sum of the E_0 's of the two ERs equals 3—the dimension of Minkowski spacetime in this case. Furthermore, such pairs are related by two Knapp–Stein integral operators, as in the $SL(2, \mathbb{R})$ case, (though, some of those operators that act from the ‘-’ ER to the ‘+’ ER degenerate into differential operators, see below). Thus, each doublet can be represented pictorially by any such \pm pair from the quartet.

The singlets are $\chi_n^s = \left[\frac{3}{2}, n - \frac{1}{2} \right], n \in \mathbb{N}$, and the expression for their signatures can be obtained from the signatures of $\chi_{q,k}^\pm$ by setting $q = 2n, k = -\frac{1}{2}q = -n$ (then \pm coincide)⁸.

In table 1 we give the Harish-Chandra parameters m_α for all representations that we discuss in this section.

According to the results of Harish-Chandra the holomorphic discrete series happen when the numbers m_α are negative integers for all non-compact roots. Thus, we see from the table that the holomorphic discrete series are contained in the ERs $\chi_{q,k}^+$.

The limits of the holomorphic discrete series happen when some of the non-compact numbers m_α become zero, while the rest of the non-compact numbers m_α remain negative. We see that these limits are contained in the ERs χ_q^+ (from the doublets).

⁸ Obviously the doublets and singlets are not related to any finite-dimensional representations.

5.2. Holomorphic discrete series and lowest weight representations

Next we discuss how the lowest weight positive energy representations fit in the multiplets and when they are infinitesimally equivalent to holomorphic discrete series.

We are interested in the positive energy UIRs of \mathcal{G} which are given as follows [30–32] (with $s_0 \in \frac{1}{2}\mathbb{Z}_+$):⁹

$$\text{Rac} : D(E_0, s_0) = D(1/2, 0), \quad \text{Di} : D(E_0, s_0) = D(1, 1/2), \quad (5.4a)$$

$$D(E_0 \geq 1, s_0 = 0), \quad D(E_0 \geq 3/2, s_0 = 1/2), \quad D(E_0 \geq s_0 + 1, s_0 \geq 1). \quad (5.4b)$$

The UIRs in (5.4a) are the two singleton representations discovered by Dirac [30] and the last case in (5.4b) corresponds to the spin- s_0 massless representations [31]. We note that for these UIRs m_2 is never a positive integer, m_3 is a positive integer only for $E_0 = 1/2, 1$, in which case $m_3 = 2, 1$, (Rac, Di), respectively. Similarly, m_4 is a positive integer only for $E_0 - s_0 = 1$, and that integer is $m_4 = 1$.

We want to see which positive energy irreps would fit in our multiplets. First we note that no such irreps can fit $\chi_{q,k}^-$ since $E_0 = \frac{3}{2} - \frac{1}{2}q - k \leq 0$, and $\chi'_{q,k}$ since $E_0 = \frac{3}{2} - \frac{1}{2}q \leq 1$ but $s_0 = \frac{1}{2}(q - 1) + k \geq 1$.

We note that all positive energy irreps from (5.4b) that would fit some multiplet can be parametrized with one parameter $k \in \mathbb{Z}, k \geq -1$. The parametrization is as follows:

$$m_1 = q = 2s_0 + 1, \quad m_2 = -q - k \implies E_0 = 2 + s_0 + k$$

In fact, for $k \geq 1$ we obtain tautologically all ERs $\chi_{q,k}^+, q = 2s_0 + 1$ (cf the table) which contain all holomorphic discrete series. For $k = 0$ we obtain all ERs $\chi_q^+, q = 2s_0 + 1$, (cf the table), which contain all limits of holomorphic discrete series. Finally, for $k = -1$ we obtain the positive energy irreps $D(1, 0), D(\frac{3}{2}, \frac{1}{2}), D(s_0 + 1, s_0)$ ($s_0 \geq 1$), which are contained in the ERs $\chi_1^-, \chi_1^s, \chi_{2s_0-1,1}^+$.

From this we see that the above would fit the EHW [4] picture with $A(\lambda_0) = 1$ (the parameter z from [4] corresponds to our $-k$). Accordingly the irreps $D(1, 0), D(\frac{3}{2}, \frac{1}{2}), D(s_0 + 1, s_0)$ ($s_0 \geq 1$), are in GVMs which are FRPs. In fact, exceptionally, the UIRs $D(1, 0), D(\frac{3}{2}, \frac{1}{2})$, are isomorphic to the corresponding GVMs since they happen to be irreducible¹⁰.

Finally, from the list of positive energy irreps, it remains to discuss the Rac and the Di from (5.4a). They are found in doublets. With respect to the positive energy spectrum they are isolated points below—by $\frac{1}{2}$ -spacing—the FRPs $D(1, 0), D(\frac{3}{2}, \frac{1}{2})$, respectively.

The Rac is in the reducible ER $\chi_{1,\frac{1}{2}}^-$. This ER is partially equivalent to the ER $\chi_{1,\frac{1}{2}}^+$ (denoted in the table as Rac*). The latter's GVM is irreducible and also of positive energy: $s_0 = 0, E_0 = \frac{5}{2}$. The intertwining operator acting from the Rac ER to Rac* is the d'Alembert operator (obtained by reduction of a Knapp–Stein operator [22]).

The Di is in the reducible ER $\chi_{2,-\frac{1}{2}}^-$. This ER is partially equivalent to the ER $\chi_{2,-\frac{1}{2}}^+$ (denoted in the table as Di*). The latter's GVM is irreducible and also of positive energy: $s_0 = \frac{1}{2}, E_0 = 2$.

All this is illustrated in figures 3 and 4.

⁹ We have adopted the notation of [31], so that $D(E_0, s_0)$ is the UIR which is contained as subrepresentation of the ER/GVM with signature $\chi = [E_0, s_0]$.

¹⁰ Brief explanation what happens to the case $D(1, 0)$. In this case the corresponding Verma module is reducible under roots $\alpha_1, \alpha_3, \alpha_4$, ($m_1 = m_3 = m_4 = 1$), however, the singular vectors corresponding to the non-compact roots α_3, α_4 turn out to be descendants of the singular vector corresponding to the compact root α_1 . Consequently, when factorizing the Verma module the resulting GVM is irreducible. The other case is similar, see details in [22].

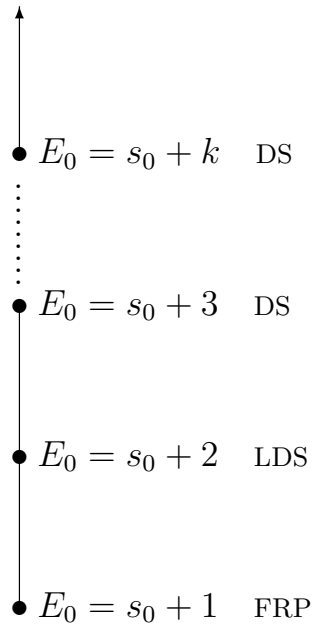


Figure 3. $so(3, 2)$, $s_0 = 1, 3/2, \dots$

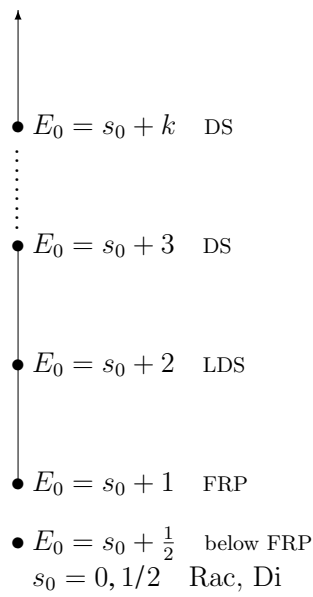


Figure 4. $so(3, 2)$, $s_0 = 0, 1/2$.

6. $SU(2, 2)$

In the 4D conformal case the ERs/GVMs that have finite \mathcal{K} -type representations can be induced from the maximal parabolic subalgebra $\mathcal{P}_{\max} = \mathcal{M}_2 \mathcal{A}_2 \mathcal{N}_2$, where $\mathcal{M}_2 = so(3, 1)$,

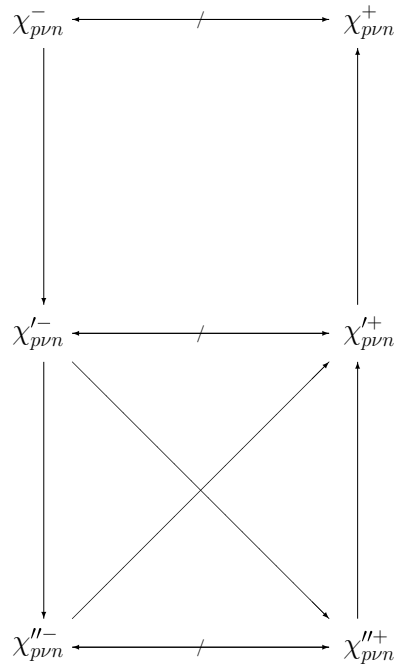


Figure 5. Sextet of partially equivalent ERs and intertwining operators for $so(4, 2)$ and $so(5, 1)$ (arrows are differential operators, dashed arrows are integral operators).

$\dim \mathcal{A}_2 = 1, \dim \mathcal{N}_2 = 4$. The signatures are given by $\chi = [j_1, j_2; d]$, where j_1, j_2 parametrize the finite-dimensional irreps of $so(3, 1)$ and d is the energy or conformal weight (parametrizing the \mathcal{A}_2 -character)¹¹.

Alternatively, the ERs/GVMs are determined by the three Dynkin labels: $m_i = \langle \Lambda + \rho, \alpha_i \rangle$, where α_i are the simple roots. The relation between the two parametrizations is

$$\begin{aligned} m_1 &= 2j_1 + 1, & m_2 &= 1 - d - j_1 - j_2, & m_3 &= 2j_2 + 1; \\ \chi &= [j_1, j_2; d] = \left[\frac{1}{2}(m_1 - 1), \frac{1}{2}(m_3 - 1); 2 - m_2 - \frac{1}{2}(m_1 + m_3) \right]. \end{aligned} \tag{6.1}$$

The numbers $m_{12} = m_1 + m_2, m_{23} = m_2 + m_3, m_{13} = m_1 + m_2 + m_3$, correspond to the three non-simple roots in an obvious manner. The roots α_1, α_3 are compact, the other—non-compact. The set of the six numbers m_α are the Harish-Chandra parameters.

6.1. Multiplets, finite-dimensional irreps and discrete series

The reducible ERs/GVMs are grouped in sextets, doublets and singlets [33, 34]. The sextets are depicted in figure 5, cf [33, 34]. The signatures of the ERs/GVMs of the sextet are

¹¹ The ERs of the group $G = SU(2, 2)$ are induced from the parabolic subgroup $P_2 = M_2 A_2 N_2$, cf (2.1), and the signature of $M_2 \cong SL(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ is given by (j_1, j_2) parametrizing the irreps of $sl(2, \mathbb{C}) \cong so(3, 1)$ and the signature $\epsilon = 0, 1$ parametrizing the character of \mathbb{Z}_2 . In what follows we shall suppress ϵ since in the multiplet points (see below) it is fixed: $\epsilon = (p + \nu + n - 1) \pmod{2}$, and thus, we shall consider only the Lie algebra picture.

given by

$$\begin{aligned}
 \chi_{pvn}^- &= \left[\frac{1}{2}(p-1), \frac{1}{2}(n-1); 2-\nu-\frac{1}{2}(p+n) \right] \\
 \chi_{pvn}^+ &= \left[\frac{1}{2}(n-1), \frac{1}{2}(p-1); 2+\nu+\frac{1}{2}(p+n) \right] \\
 \chi_{pvn}'^- &= \left[\frac{1}{2}(p+\nu-1), \frac{1}{2}(n+\nu-1); 2-\frac{1}{2}(p+n) \right] \\
 \chi_{pvn}'^+ &= \left[\frac{1}{2}(n+\nu-1), \frac{1}{2}(p+\nu-1); 2+\frac{1}{2}(p+n) \right] \\
 \chi_{pvn}''^- &= \left[\frac{1}{2}(\nu-1), \frac{1}{2}(p+n+\nu-1); 2+\frac{1}{2}(p-n) \right] \\
 \chi_{pvn}''^+ &= \left[\frac{1}{2}(p+n+\nu-1), \frac{1}{2}(\nu-1); 2+\frac{1}{2}(n-p) \right].
 \end{aligned} \tag{6.2}$$

The sextets are in one-to-one correspondence with the finite-dimensional irreps of G since in each sextet there is exactly one ER/GVM which contains (as subrepresentation) a finite-dimensional irrep. The latter are parametrized by the positive integer Dynkin labels which we denote as in [35] by $p, \nu, n \in \mathbb{N}$. Correspondingly, the finite-dimensional irrep is denoted E_{pvn} , it has dimension: $p\nu n(p+\nu)(n+\nu)(p+\nu+n)/12$, [34], and is contained in the ER/GVM denoted by χ_{pvn}^- .

As a consequence, the sextets also hold the discrete series representations. For fixed p, ν, n the six representations of the sextet are denoted by $\chi_{pvn}^\pm, \chi_{pvn}'^\pm, \chi_{pvn}''^\pm$. The discrete series are contained in $\chi_{pvn}^+, \chi_{pvn}'^+, \chi_{pvn}''^+$ —we know that there are three distinct non-conjugate cases of discrete series [6] (the two cases $\chi_{pvn}''^\pm$ are conjugate and count as one case).

The limits of discrete series representations are in some doublets. The doublets are of three kinds, denoted by ${}^1\chi_{p\nu}^\pm, {}^2\chi_{pn}^\pm, {}^3\chi_{\nu n}^\pm$, and the expression for their signatures can be obtained from the signatures of χ_{pvn}^\pm , by setting, $n = 0, \nu = 0, p = 0$, respectively, i.e.,

$$\begin{aligned}
 {}^1\chi_{p\nu}'^- &= \left[\frac{1}{2}(p+\nu-1), \frac{1}{2}(\nu-1); 2-\frac{1}{2}p \right] \\
 {}^1\chi_{p\nu}'^+ &= \left[\frac{1}{2}(\nu-1), \frac{1}{2}(p+\nu-1); 2+\frac{1}{2}p \right] \\
 {}^2\chi_{pn}'^- &= \left[\frac{1}{2}(p-1), \frac{1}{2}(n-1); 2-\frac{1}{2}(p+n) \right] \\
 {}^2\chi_{pn}'^+ &= \left[\frac{1}{2}(n-1), \frac{1}{2}(p-1); 2+\frac{1}{2}(p+n) \right] \\
 {}^3\chi_{\nu n}'^- &= \left[\frac{1}{2}(\nu-1), \frac{1}{2}(n+\nu-1); 2-\frac{1}{2}n \right] \\
 {}^3\chi_{\nu n}'^+ &= \left[\frac{1}{2}(n+\nu-1), \frac{1}{2}(\nu-1); 2+\frac{1}{2}n \right].
 \end{aligned} \tag{6.3}$$

Note that for all pairs of ERs/GVMs with signature distinguished by \pm the sum of the d 's of the two ERs/GVMs equals 4—the dimension of Minkowski spacetime in this case. Furthermore, the ERs of such pairs are related by two Knapp–Stein integral operators [25].

Finally the singlets are denoted by $\chi_\nu^s = \left[\frac{1}{2}(\nu-1), \frac{1}{2}(\nu-1); 2 \right]$, ($\nu \in \mathbb{N}$), and the expression for their signatures can be obtained from the signatures of χ_{pvn}^\pm , by setting, $n = 0$ and $p = 0$ (then \pm coincide)¹².

In table 2 we give the Harish-Chandra parameters m_α for all representations that we discuss in this section¹³.

According to the results of Harish-Chandra the holomorphic discrete series happen when the numbers m_α are negative integers for all non-compact roots. Thus, we see from the table that the holomorphic discrete series are contained in the ERs χ_{pvn}^+ . The limits of the holomorphic discrete series happens when some of the non-compact numbers m_α become zero, while the rest of the non-compact numbers m_α remain negative. We see that these limits are contained in the ERs ${}^2\chi_{pn}^+$ (also obtained from χ_{pvn}^+ for $\nu = 0$).

¹² Obviously the doublets and singlets are not related to any finite-dimensional representations.

¹³ Matters are arranged as discussed so that the Dynkin labels are equal to p, ν, n for the series χ_{pvn}^- (which contains the finite-dimensional irreps).

Table 2. Harish-Chandra parameters for $so(4, 2)$: $m_i = \langle \Lambda + \rho, \alpha_i \rangle, i = 1, 2, 3, m_{12} = m_1 + m_2, m_{23} = m_2 + m_3, m_{13} = m_1 + m_2 + m_3.$

| ER | m_1 | m_2 | m_3 | m_{12} | m_{23} | m_{13} |
|------------------------------|----------------------------|--------------|----------------------------|----------------------------|-----------------------------|-------------|
| χ_{pvn}^- | p | v | n | $p + v$ | $n + v$ | $p + n + v$ |
| χ_{pvn}^+ | n | $-p - v - n$ | p | $-p - v$ | $-n - v$ | $-v$ |
| $\chi_{pvn}^{\prime-}$ | $p + v$ | $-v$ | $n + v$ | p | n | $p + n + v$ |
| $\chi_{pvn}^{\prime+}$ | $n + v$ | $-p - v - n$ | $p + v$ | $-p$ | $-n$ | v |
| $\chi_{pvn}^{\prime\prime-}$ | v | $-v - p$ | $p + n + v$ | $-p$ | n | $n + v$ |
| $\chi_{pvn}^{\prime\prime+}$ | $p + n + v$ | $-n - v$ | v | p | $-n$ | $p + v$ |
| $^1\chi_{pv}^-$ | $p + v$ | $-v$ | v | p | 0 | $p + v$ |
| $^1\chi_{pv}^+$ | v | $-p - v$ | $p + v$ | $-p$ | 0 | v |
| $^2\chi_{pn}^-$ | p | 0 | n | p | n | $p + n$ |
| $^2\chi_{pn}^+$ | n | $-p - n$ | p | $-p$ | $-n$ | 0 |
| $^3\chi_{vn}^-$ | v | $-v$ | $n + v$ | 0 | n | $n + v$ |
| $^3\chi_{vn}^+$ | $n + v$ | $-v - n$ | v | 0 | $-n$ | v |
| χ_v^3 | v | $-v$ | v | 0 | 0 | v |
| $P_{n',k,\epsilon,v'}^1$ | $\frac{1}{2}(k - v' + n')$ | $-k$ | $\frac{1}{2}(k - v' - n')$ | $\frac{1}{2}(n' - k - v')$ | $-\frac{1}{2}(k + v' + n')$ | $-v'$ |

6.2. Holomorphic discrete series and lowest weight representations

Next we discuss how the lowest weight positive energy representations fit in the multiplets, and when they are infinitesimally equivalent to holomorphic discrete series.

There are two basic cases of positive energy representations [23]:

$$(1) \quad j_1 j_2 \neq 0, \quad (2) \quad j_1 j_2 = 0.$$

In case (1) the positive energy representations fulfil the condition [23]:

$$d \geq 2 + j_1 + j_2, \quad j_1 j_2 \neq 0.$$

For $d > 2 + j_1 + j_2$ the GVMs are irreducible and unitary.

The point $d = d_0 \equiv 2 + j_1 + j_2$ is the first reduction point. In our picture it is realized in the GVM with signature $\chi_{p1n}^{\prime+}$, so that $j_1 = \frac{1}{2}n, j_2 = \frac{1}{2}p$.

The point $d = d_0 + 1$ is a limit of discrete series, while the integer points with $d \geq d_0 + 2$ are the holomorphic discrete series. Indeed, the former are contained in $^2\chi_{pn}^+$, while the latter are contained in the ERs with signature χ_{pvn}^+ . In both cases we have $j_1 = \frac{1}{2}(n - 1), j_2 = \frac{1}{2}(p - 1), d = 2 + v + \frac{1}{2}(n + p)$, where $n, p > 1$ and $v = 0, v \in \mathbb{N}$, distinguishes the two cases.

Thus, these cases correspond to $c_0 = 1$ (see above) and $A(\lambda_0) = 1$ in the terminology of [4]. Here and below the unitarity parameter z of [4] is related to ours as $z = -d + d_0 + A(\lambda_0)$.

In case (2) the positive energy representations fulfil the condition [23]:

$$d \geq 1 + j_1 + j_2, \quad j_1 j_2 = 0.$$

For $d > 1 + j_1 + j_2$ the GVMs are irreducible and unitary.

The point $d_0^0 = 1 + j_1 + j_2$ is the first reduction point. These are the *massless* representations of $so(4, 2)$.

For $j_1 + j_2 \geq 1$ the FRP is realized in the ERs/GVMs with signatures: $\chi_{11n}^{\prime\prime+}$, with $j_1 = \frac{1}{2}(n + 1) \geq 1, j_2 = 0$ and $\chi_{p11}^{\prime\prime-}$, with $j_1 = 0, j_2 = \frac{1}{2}(p + 1) \geq 1$.

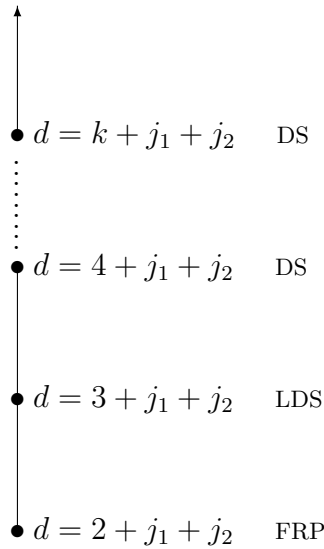


Figure 6. $so(4, 2)$, $j_1 j_2 \neq 0$.

For $j_1 + j_2 = \frac{1}{2}$ the FRP is realized in the ERs/GVMs with signatures: ${}^3\chi_{11}^-$, with $j_1 = \frac{1}{2}$, $j_2 = 0$, and ${}^1\chi_{11}^-$, with $j_1 = 0$, $j_2 = \frac{1}{2}$.¹⁴

For $j_1 = j_2 = 0$ the FRP is realized in the ER with signature: ${}^2\chi_{11}^-$.¹⁵

As we shall see, these cases correspond to $c_0 = 1$ (see above) and $A(\lambda_0) = 2$ in the terminology of [4] (the FRP is $z = A(\lambda_0) = 2$).

The point next to the FRP ($z = 1$ by [4]) with $d = d_0^0 + 1 = 2 + j_1 + j_2$ is part of the analytic continuation of the discrete series.

For $j_1 + j_2 \geq \frac{1}{2}$ it fits the ERs ${}^3\chi_{1n}^+$, so that $j_1 = \frac{1}{2}n$, $j_2 = 0$, and ${}^1\chi_{p1}^+$, so that $j_1 = 0$, $j_2 = \frac{1}{2}p$.

For $j_1 = j_2 = 0$ it is realized in the singlet ER with signature: χ_1^s .

The next point with $d = d_0^0 + 2 = 3 + j_1 + j_2$ ($z = 0$ by [4]) fits the ERs ${}^2\chi_{pn}^+$ with either $p = 1$ or $n = 1$, which contain limits of discrete series (with $j_1 = \frac{1}{2}(n - 1)$, $j_2 = \frac{1}{2}(p - 1)$, as above for $j_1 j_2 \neq 0$).

Finally, the cases with integer $d \geq d_0^0 + 3 = 4 + j_1 + j_2$ ($z < 0$ by [4]) are realized by the ERs χ_{pvn}^+ which contain the holomorphic discrete series (as above for $j_1 j_2 \neq 0$).

All this is illustrated in figures 6 and 7.

6.3. Induction from another parabolic

The ERs discussed until now can also be induced from the minimal parabolic subgroup as shown in [37], however, then they appear in larger multiplets [34]. In particular, the sextets

¹⁴ The two conjugated representations are two-component massless Weyl spinors. They are partially equivalent to the ERs ${}^3\chi_{11}^+$, ${}^1\chi_{11}^+$ mentioned below, and the corresponding Knapp–Stein operators from these FRPs degenerate to the two well-known first order conjugated Weyl equations.

¹⁵ This massless scalar representation is partially equivalent to the scalar ER ${}^2\chi_{11}^+$, mentioned below. The two ERs are related by Knapp–Stein integral operators, however, the operator from ${}^2\chi_{11}^-$ to ${}^2\chi_{11}^+$ degenerates to the d’Alembert operator [36].

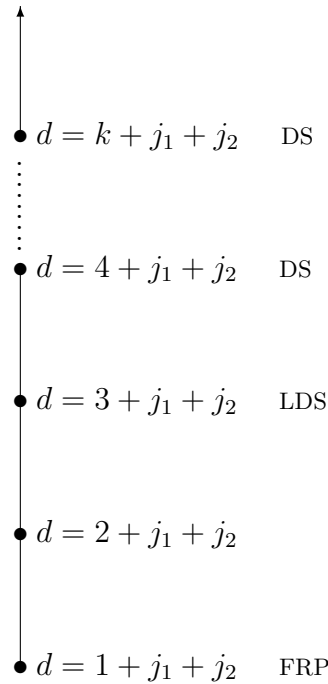


Figure 7. $so(4, 2)$, $j_1 j_2 = 0$.

are part of 24 plets, where each ER corresponds to an element of the Weyl group W of $\mathcal{G}^{\mathbb{C}} = so(6, \mathbb{C})$ (recall that $|W| = 24$).

But besides the minimal and maximal non-cuspidal parabolics (of dimensions 9 and 11, respectively), the group $SU(2, 2)$ has a maximal cuspidal parabolic of dimension 10: $P_1 = M_1 A_1 N_1$, where $M_1 = SL(2, \mathbb{R}) \times SO(2)$, $\dim A_1 = 1$, $\dim N_1 = 5$. The signatures here are $\chi_1 = [n', k, \epsilon, v']$, where $n' \in \mathbb{Z}$ is a character of $SO(2)$, $v' \in \mathbb{C}$ is a character of A_1 , k, ϵ fix a discrete series representation of $SL(2, \mathbb{R})$, $k \in \mathbb{N}$, $\epsilon = 0, 1$, or a limit thereof when $k = 0$. In the integer points we are interested in $\epsilon = k \pmod{2}$, $v' \in \mathbb{Z}$, and the relation with the Dynkin labels is as follows [34]:

$$m_1 = \frac{1}{2}(k - v' + n'), \quad m_2 = -k, \quad m_3 = \frac{1}{2}(k - v' - n') \quad (6.4)$$

Clearly, representations induced in this way will not describe all ERs with finite K-type, cf the table. For instance, one cannot obtain the ERs of type χ_{pvn}^- . Further, there are some restrictions on the values of n', k, v' when matching the other five series, e.g., to have finite K-types must hold the condition: $k > v' + |n'|$. Furthermore, in order to fit the holomorphic discrete series, i.e., χ_{pvn}^+ , must hold: $k > |v'| + |n'|$. These representations describe the limits of discrete series when $v' = 0$ and $k > |n'|$.

7. Outlook

In the present paper we restricted to the conformal group case. Similar explicit descriptions can be easily achieved for the other non-compact groups with lowest/highest weight

representations. We also plan to extend these considerations [38] to the supersymmetric cases using previous results on the classification of positive energy irreps in various dimensions [39–42], and also to the quantum group setting using [43]. Such considerations are expected to be very useful for applications to string theory and integrable models, cf., e.g., [44].

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